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LOCALIZATION FOR THE TORSION FUNCTION AND THE STRONG HARDY INEQUALITY

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Abstract. Two-sided bounds for the efficiency of the torsion function are obtained in terms of the square of the distance to the boundary function under the hypothesis that the Dirichlet Laplacian satisfies a strong Hardy inequality. Localization properties of the torsion function are obtained under that hypothesis. An example is analyzed in detail.

§1. *Introduction and main results.* Let Ω be an open set in $\mathbb{R}^m$, $m \geqslant 1$, with finite Lebesgue measure $|\Omega|$, $0 < |\Omega| < \infty$, and with boundary $\partial\Omega$. The torsion function for Ω is the unique solution of

$$-\Delta v = 1, \qquad v \in H_0^1(\Omega),$$

and is denoted by v_Ω. The function v_Ω is non-negative and satisfies,

$$\lambda_1(\Omega)^{-1} \leqslant \|v_\Omega\|_\infty \leqslant (4 + 3m \log 2)\lambda_1(\Omega)^{-1}, \tag{1}$$

where

$$\lambda_1(\Omega) = \inf_{\varphi \in H_0^1(\Omega)\setminus\{0\}} \frac{\|\nabla\varphi\|_2^2}{\|\varphi\|_2^2} \tag{2}$$

is the first eigenvalue of the Dirichlet Laplacian and where $\|\cdot\|_p$ denotes the standard L^p norm, $1 \leqslant p \leqslant \infty$. The m-dependent constant in the right-hand side of (1) has subsequently been improved [**17, 24**]. We denote the sharp constant by $\mathfrak{c}_m$,

$$\mathfrak{c}_m = \sup\{\lambda_1(\Omega)\|v_\Omega\|_\infty : \Omega \text{ open in } \mathbb{R}^m,\ 0 < |\Omega| < \infty\}. \tag{3}$$

The torsion function and its L^1 norm, the torsional rigidity, play key roles in different parts of analysis. For example, the torsional rigidity of a cross section of a beam appears in the computation of the angular change when a beam of a given length and a given modulus of rigidity is exposed to a twisting moment [**4, 22**]. It also arises in the definition of gamma convergence [**10**] and in the study of minimal submanifolds [**20**]. Moreover, $v_\Omega(x)$ equals the expected lifetime of Brownian motion in Ω starting at $x \in \Omega$. This immediately implies the non-negativity of v_Ω, and that for open sets Ω_1, Ω_2,

$$\Omega_1 \subset \Omega_2 \Rightarrow v_{\Omega_1}(x) \leqslant v_{\Omega_2}(x),\ \forall x \in \Omega_1. \tag{4}$$

This in turn implies that the torsional rigidity $\|v_\Omega\|_1$ is monotone increasing in Ω. The torsion function has been studied extensively, and numerous works have been written on this subject. We just mention the paper [**9**], and the references therein.

The torsion function is defined for an open subset $\Omega \subset \mathbb{R}^m$ of infinite measure, provided that the bottom of the spectrum of the Dirichlet Laplacian, denoted by $\lambda_1(\Omega)$, is bounded away from 0. Indeed, by considering an increasing sequence of sets $\Omega \cap B(0; k)$, $k \in \mathbb{N}$,

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where $B(p; R) = \{x \in \mathbb{R}^m : |p - x| < R\}$ denotes the open ball with center p and radius R, one obtains a weak solution of

$$-\Delta v_\Omega = 1, \qquad v|_{\partial\Omega} = 0. \tag{5}$$

This solution is non-negative, satisfies both (4) and (1), and satisfies the probabilistic interpretation mentioned above. See [7, Theorem 1], [5, Theorem 5.3], and [10] for further details.

We remark that the torsion function has been defined in greater generality. Instead of the Dirichlet Laplacian, one considers a positive, self-adjoint Schrödinger operator $L := -\Delta + V$, where V is bounded, measurable, and non-negative (or even more generally, a positive, self-adjoint, elliptic operator of second order) acting in $L^2(\Omega)$ with Dirichlet or Neumann boundary conditions or on closed manifolds. In the latter situation, one requires potentials V such that the bottom of the spectrum of the Laplacian is bounded away from 0. We denote by v_L the torsion function associated to L. It was discovered in recent papers [2, 3], and references therein that under appropriate conditions, the reciprocal v_L^{-1} of the torsion function v_L can be used for approximating eigenvalues and eigenfunctions of the Schrödinger operator L. The phenomenon of localization of eigenfunctions of Schrödinger operators is a prominent and very active research area and has important applications in the applied sciences. The literature is extensive. See, for example, the review paper of [19]. Explicit estimates on the localization of the eigenfunctions in terms of bounds on their exponential decay away from the subdomains, where they concentrate, were obtained in [3].

In this paper we consider the *κ-localization*, the *localization*, and the *efficiency* of the torsion function of the Dirichlet Laplacian on an open set Ω in $\mathbb{R}^m$ with $0 < |\Omega| < \infty$—see Definition 2 and Definition 3 below. These notions have been first introduced for Dirichlet eigenfunctions of Schrödinger operators and go back to [19] and, respectively, [21, 23]. The notion of efficiency, defined as the mean to max ratio, can be viewed as a rough measure of localization.

In [6], we studied the efficiency and localization for the torsion function of a Schrödinger operator $-\Delta + V$ acting in $L^2(\Omega)$ with Dirichlet boundary condition. Among other results it was shown, in Theorem 4, that under appropriate conditions, localization for the torsion function implies localization for the first Dirichlet eigenfunction. The converse does not hold. Consider for example a sequence of ellipsoids Ω_n with semi-axes of length 1 and n respectively. We see by Theorem 1(iii) below that $((v_{\Omega_n}))$ does not localize. On the other hand, it was shown in [8, Example 10] that the corresponding sequence of first Dirichlet eigenfunctions localizes. Unlike the torsion function, the first Dirichlet eigenfunction is not monotone on set inclusion, and this in general complicates its analysis.

In this paper, we continue this study. Our results are obtained under the hypothesis that the Dirichlet Laplacian satisfies the strong Hardy inequality, defined as follows.

Definition 1. The Dirichlet Laplacian $-\Delta$ acting in $L^2(\Omega)$ satisfies the strong Hardy inequality, with constant $c_\Omega \in (0, \infty)$, if

$$\|\nabla w\|_2^2 \geqslant \frac{1}{c_\Omega} \int_\Omega \frac{w^2}{d_\Omega^2}, \quad \forall\, w \in C_c^\infty(\Omega), \tag{6}$$

where d_Ω is the distance to the boundary function,

$$d_\Omega(x) = \inf\{|x - y| : y \in \mathbb{R}^m \setminus \Omega\}, \qquad x \in \Omega.$$

Both the validity and applications of inequalities like (6) to spectral theory and partial differential equations have been investigated in depth. See, for example, [1, 11–14]. In particular, it was shown in [1, p. 208], that for any proper simply connected open subset Ω in $\mathbb{R}^2$, $c_\Omega = 16$.

Our first result in Theorem 1, stated below, states that the efficiency of the torsion function of the Dirichlet Laplacian on an open set Ω of $\mathbb{R}^m$ with $0 < |\Omega| < \infty$ can be bounded from above and below by the efficiency of the square of the distance to the boundary function, whereas item (ii) and (iii) of Theorem 1 give sufficient conditions for the localization of the torsion function to hold and, respectively, not to hold.

In Theorem 2(i)–(iii) we analyze an example illustrating the use of Theorem 1. In Theorem 2(iv) we give an example of κ-localization with $0 < \kappa < 1$.

We now define the notion of efficiency and localization in precise terms. The efficiency of v_Ω, introduced in [15], is defined as follows.

Definition 2. Let Ω be an open set in $\mathbb{R}^m$ with $0 < |\Omega| < \infty$. The efficiency, or mean to max ratio, of v_Ω is

$$\Phi(\Omega) = \frac{\|v_\Omega\|_1}{|\Omega| \|v_\Omega\|_\infty}.$$

Let (Ω_n) be a sequence of open sets in $\mathbb{R}^m$ with $0 < |\Omega_n| < \infty$, $n \in \mathbb{N}$. We say that (v_{Ω_n}) has vanishing efficiency if $\lim_{n\to\infty} \Phi(\Omega_n) = 0$.

The following notion of localization for the torsion function has been motivated by the one for eigenfunctions in [19], and has been used previously in [6].

Definition 3. For any sequence of open sets (Ω_n) in $\mathbb{R}^m$ with $0 < |\Omega_n| < \infty$, $n \in \mathbb{N}$, let

$$\mathfrak{A}((\Omega_n)) = \left\{(A_n) : (\forall n \in \mathbb{N})(A_n \subset \Omega_n, A_n \text{ measurable}),\ \lim_{n\to\infty} \frac{|A_n|}{|\Omega_n|} = 0\right\}, \tag{7}$$

and

$$\kappa = \sup\left\{\limsup_{n\to\infty} \frac{\int_{A_n} v_{\Omega_n}}{\|v_{\Omega_n}\|_1} : (A_n) \in \mathfrak{A}((\Omega_n))\right\}. \tag{8}$$

We say that if (i) $0 < \kappa < 1$, then (v_{Ω_n}) κ-localizes, (ii) $\kappa = 1$, then (v_{Ω_n}) *localizes*, and (iii) $\kappa = 0$, then (v_{Ω_n}) *does not localize*.

Before stating our main results we review some basic facts. We note that

$$B(x; d_\Omega(x)) \subset \Omega, \tag{9}$$

and $|B(x; d_\Omega(x))| \leqslant |\Omega|$. Hence d_Ω is bounded from above and

$$\|d_\Omega\|_\infty \leqslant \left(\omega_m^{-1}|\Omega|\right)^{1/m},$$

where $\omega_m = |B(0; 1)|$. Since the torsion function is pointwise increasing with respect to the domain,(4), it follows by (9) that,

$$v_\Omega(x) \geqslant v_{B(x;d_\Omega(x))}(x) = \frac{d_\Omega(x)^2}{2m}, \quad \forall x \in \Omega, \tag{10}$$

where we have used that

$$v_{B(p;R)}(x) = \frac{1}{2m}(R^2 - |x-p|^2), \quad \forall x \in B(p; R).$$

By (10),

$$\frac{1}{2m}\|d_\Omega^2\|_1 \leqslant \|v_\Omega\|_1 \,. \tag{11}$$

Under the additional assumption that Ω satisfies (6), it was shown in [7, Theorem 2] that

$$\|v_\Omega\|_1 \leqslant c_\Omega \|d_\Omega^2\|_1. \tag{12}$$

Furthermore, d_Ω is uniformly continuous. Hence for any $\eta \in \mathbb{R}^+$, the subset $\{d_\Omega \geqslant \eta\}$ is relatively closed in Ω, and hence measurable.

Throughout we denote for $\nu \geqslant 0$, by j_ν the first positive zero of the Bessel function J_ν. For the ball $B(p;1) \subset \mathbb{R}^m$, $m \geqslant 2$, we have $\lambda_1(B(p;1)) = j^2_{(m-2)/2}$. Our main results are as follows.

THEOREM 1. (i) *If Ω is an open subset of $\mathbb{R}^m$ with $0 < |\Omega| < \infty$, and which satisfies (6) with the strong Hardy constant c_Ω, then*

$$(2m\mathfrak{c}_m c_\Omega)^{-1}\frac{\|d_\Omega^2\|_1}{|\Omega|\|d_\Omega^2\|_\infty} \leqslant \Phi(\Omega) \leqslant c_\Omega j^2_{(m-2)/2}\frac{\|d_\Omega^2\|_1}{|\Omega|\|d_\Omega^2\|_\infty}, \tag{13}$$

where $\mathfrak{c}_m$ is the sharp constant in (3).

(ii) *Let (Ω_n) be a sequence of open sets in $\mathbb{R}^m$ with $0 < |\Omega_n| < \infty$, $n \in \mathbb{N}$, and which satisfies (6) with strong Hardy constants c_{Ω_n}. Suppose*

$$c = \sup\{c_{\Omega_n} : n \in \mathbb{N}\} < \infty. \tag{14}$$

If (η_n) is a sequence of strictly positive real numbers such that

$$\lim_{n\to\infty}\frac{|\{d_{\Omega_n} \geqslant \eta_n\}|}{|\Omega_n|} = 0, \tag{15}$$

and

$$\lim_{n\to\infty}\frac{\eta_n^2|\Omega_n|}{\int_{\{d_{\Omega_n}\geqslant \eta_n\}} d_{\Omega_n}^2} = 0, \tag{16}$$

then (v_{Ω_n}) localizes along the sequence $A_n = \{x \in \Omega_n : d_{\Omega_n} \geqslant \eta_n\}$.

(iii) *Let (Ω_n) be a sequence of open sets in $\mathbb{R}^m$ with $0 < |\Omega_n| < \infty$, $n \in \mathbb{N}$, which satisfies (6) with strong Hardy constants c_{Ω_n}. Suppose that (14) holds. If any sequence (A_n) of measurable sets, $A_n \subset \Omega_n$, $n \in \mathbb{N}$, with*

$$\lim_{n\to\infty}\frac{|A_n|}{|\Omega_n|} = 0,$$

satisfies

$$\lim_{n\to\infty}\frac{\int_{A_n} d_{\Omega_n}^2}{\int_{\Omega_n} d_{\Omega_n}^2} = 0,$$

then (v_{Ω_n}) does not localize.

Theorem 1(i) can be interpreted as follows. Given an open subset Ω of $\mathbb{R}^m$ with $0 < |\Omega| < \infty$, we define the efficiency of d_Ω^2 as

$$D(\Omega) = \frac{\|d_\Omega^2\|_1}{|\Omega|\|d_\Omega^2\|_\infty}\,.$$

Theorem 1(i) asserts that under condition (6), the efficiencies of v_Ω and d_Ω^2 are comparable.

Let (Ω_n) be a sequence of open sets in $\mathbb{R}^m$ with $0 < |\Omega_n| < \infty$. We say that $(d^2_{\Omega_n})$ localizes if

$$\sup\left\{\limsup_{n\to\infty} \frac{\int_{A_n} d^2_{\Omega_n}}{\|d^2_{\Omega_n}\|_1} : (A_n) \in \mathfrak{A}((\Omega_n))\right\} = 1,$$

where $\mathfrak{A}((\Omega_n))$ is given by (7). Theorem 1(iii) asserts that for any sequence (Ω_n) satisfying the conditions of Theorem 1(iii), the following holds: if $(d^2_{\Omega_n})$ does not localize, then neither does (v_{Ω_n}).

In Theorem 2 below we analyze the torsion function for a sequence of open simply connected sets $(\Omega_{\varepsilon_n,n})_{n\geqslant 4} \subset \mathbb{R}^2$ with $0 < \varepsilon_n < 1$. The construction is as follows. Let Q be the open unit square in $\mathbb{R}^2$ with vertices (0,0),(1,0),(0,1),(1,1). Let $n \in \mathbb{N}$, and for any given $n \in \mathbb{N}$, let $L_1, \dots, L_{n-1}$ be the closed line segments of lengths $1 - \varepsilon_n$ with endpoints $(\frac{1}{n}, 0), (\frac{2}{n}, 0), \dots, (\frac{n-1}{n}, 0)$ pointing in the direction (0,1). Let

$$\Omega_{\varepsilon_n,n} = Q \setminus \cup_{j=1}^{n-1} L_j.$$

THEOREM 2. *Let* $0 < \alpha < 1,\ c > 0$,

$$\varepsilon_n = cn^{-\alpha},$$

and

$$N_{\alpha,c} = \min\{n \in \mathbb{N} : n^\alpha \geqslant 2c,\ cn^{1-\alpha} \geqslant 2\}.$$

(*Note that* $N_{\alpha,c} \geqslant 4$.)

(i) *If* $n \geqslant N_{\alpha,c}$, *then*

$$\frac{1}{3072\mathfrak{c}_2}\left(cn^{-\alpha} + c^{-2}n^{2\alpha-2}\right) \leqslant \Phi(\Omega_{\varepsilon_n,n}) \leqslant \frac{64 j_0^2}{3}\left(cn^{-\alpha} + c^{-2}n^{2\alpha-2}\right), \tag{17}$$

where $\mathfrak{c}_2$ *is given by* (3).

(ii) *If* $0 < \alpha < \frac{2}{3}$, *then* $(v_{\Omega_{\varepsilon_n,n}})$ *localizes along the sequence*

$$A_n = \left\{x \in \Omega_{\varepsilon_n,n} : d_{\Omega_{\varepsilon_n,n}}(x) \geqslant \frac{1}{2n}\right\}. \tag{18}$$

(iii) *If* $\frac{2}{3} < \alpha < 1$, *then* $(v_{\Omega_{\varepsilon_n,n}})$ *is not localizing.*

(iv) *If* $\alpha = \frac{2}{3}$, *then* $(v_{\Omega_{\varepsilon_n,n}})$ *is* κ_c*-localizing along* (18) *with*

$$\kappa_c = \frac{c^3}{1 + c^3}. \tag{19}$$

The paper is organized as follows. The proofs of Theorems 1 and 2(i)–(iii) are given in Section 2. The proof of Theorem 2(iv) involves tools from Brownian motion and is deferred to Section 3.

§2. *Proofs of Theorems 1 and 2(i)–(iii).* *Proof of Theorem 1*(i). Since $|\Omega| < \infty$, the inradius $\|d_\Omega\|_\infty$ is finite, and by (2) and (6),

$$\begin{aligned}
\lambda_1(\Omega) &\geqslant c_\Omega^{-1} \inf_{\varphi\in H_0^1(\Omega),\ \|\varphi\|_2^2=1} \int_\Omega \frac{\varphi^2}{d_\Omega^2} \\
&\geqslant c_\Omega^{-1}\|d_\Omega\|_\infty^{-2} \inf_{\varphi\in H_0^1(\Omega),\ \|\varphi\|_2^2=1} \|\varphi\|_2^2 \\
&= c_\Omega^{-1}\|d_\Omega\|_\infty^{-2}.
\end{aligned} \tag{20}$$

By (3) and (20),

$$\|v_\Omega\|_\infty \leqslant \mathfrak{c}_m \lambda_1(\Omega)^{-1} \leqslant \mathfrak{c}_m c_\Omega \|d_\Omega\|_\infty^2. \tag{21}$$

The lower bound in (13) follows from (21) and the lower bound in (11).

Since Ω has inradius $\|d_\Omega\|_\infty$, Ω contains an open ball with radius $\|d_\Omega\|_\infty$ and center p_Ω. By the monotonicity of the Dirichlet eigenvalues

$$\lambda_1(\Omega) \leqslant \lambda_1(B(p_\Omega; \|d_\Omega\|_\infty) = j^2_{(m-2)/2}\|d_\Omega\|_\infty^{-2}. \tag{22}$$

By the first inequality in (1) and (22),

$$\|v_\Omega\|_\infty^{-1} \leqslant j^2_{(m-2)/2}\|d_\Omega\|_\infty^{-2}. \tag{23}$$

The upper bound in (13) follows from (23) and the upper bound in (12).

Proof of Theorem 1(ii). By (6), and suppressing n dependence,

$$\begin{aligned}\int_\Omega v_\Omega &= -\int_\Omega v_\Omega \Delta v_\Omega = \int_\Omega |\nabla v_\Omega|^2 \\ &\geqslant \frac{1}{c_\Omega}\int_\Omega \frac{v_\Omega^2}{d_\Omega^2} \geqslant \frac{1}{c_\Omega}\int_{\{d_\Omega<\eta\}} \frac{v_\Omega^2}{d_\Omega^2} \\ &\geqslant \frac{1}{c_\Omega \eta^2}\int_{\{d_\Omega<\eta\}} v_\Omega^2. \end{aligned}\tag{24}$$

On the other hand, by Cauchy–Schwarz,

$$\left(\int_{\{d_\Omega<\eta\}} v_\Omega\right)^2 \leqslant |\{d_\Omega < \eta\}| \int_{\{d_\Omega<\eta\}} v_\Omega^2. \tag{25}$$

Combining (24) and (25) yields,

$$\left(\int_{\{d_\Omega<\eta\}} v_\Omega\right)^2 \leqslant c_\Omega \eta^2 |\Omega| \left(\int_{\{d_\Omega<\eta\}} v_\Omega + \int_{\{d_\Omega\geqslant\eta\}} v_\Omega\right). \tag{26}$$

Solving the quadratic inequality (26) gives,

$$\begin{aligned}\int_{\{d_\Omega<\eta\}} v_\Omega &\leqslant \frac{c_\Omega \eta^2|\Omega|}{2} + \left(\frac{c_\Omega^2\eta^4|\Omega|^2}{4} + c_\Omega\eta^2|\Omega| \int_{\{d_\Omega\geqslant\eta\}} v_\Omega\right)^{1/2} \\ &\leqslant c_\Omega\eta^2|\Omega| + \left(c_\Omega\eta^2|\Omega|\int_{\{d_\Omega\geqslant\eta\}} v_\Omega\right)^{1/2}. \end{aligned}\tag{27}$$

By (10),

$$\int_{\{d_\Omega\geqslant\eta\}} v_\Omega \geqslant \frac{1}{2m}\int_{\{d_\Omega\geqslant\eta\}} d_\Omega^2. \tag{28}$$

By (27) and (28),

$$\frac{\int_{\{d_\Omega<\eta\}} v_\Omega}{\int_{\{d_\Omega\geqslant\eta\}} v_\Omega} \leqslant \frac{2mc_\Omega\eta^2|\Omega|}{\int_{\{d_\Omega\geqslant\eta\}} d_\Omega^2} + \left(\frac{2mc_\Omega\eta^2|\Omega|}{\int_{\{d_\Omega\geqslant\eta\}} d_\Omega^2}\right)^{1/2}. \tag{29}$$

To complete the proof we suppose that (Ω_n) and (η_n) are sequences satisfying (15) and (16), respectively, and that c_{Ω_n} satisfies (14). If

$$\lim_{n\to\infty} \frac{\eta_n^2|\Omega_n|}{\int_{\{d_{\Omega_n}\geqslant\eta_n\}} d_{\Omega_n}^2} = 0,$$

then by (29)

$$\lim_{n\to\infty}\frac{\int_{\{d_{\Omega_n}<\eta_n\}} v_{\Omega_n}}{\int_{\{d_{\Omega_n}\geqslant\eta_n\}} v_{\Omega_n}}=0,$$

and

$$\lim_{n\to\infty}\frac{\int_{\{d_{\Omega_n}<\eta_n\}} v_{\Omega_n}}{\int_{\Omega_n} v_{\Omega_n}}=0.$$

Hence

$$\lim_{n\to\infty}\frac{\int_{\{d_{\Omega_n}\geqslant\eta_n\}} v_{\Omega_n}}{\int_{\Omega_n} v_{\Omega_n}}=1. \tag{30}$$

Let $A_n=\{d_{\Omega_n}\geqslant\eta_n\}$. Then A_n, $n\in\mathbb{N}$, is relatively closed in Ω, and hence measurable. By hypothesis (15),

$$\lim_{n\to\infty}\frac{|A_n|}{|\Omega_n|}=0,$$

and by (30) we have $\limsup_{n\to\infty}\frac{\int_{A_n} v_{\Omega_n}}{\|v_{\Omega_n}\|_1}=1$. Hence (8) holds with $\kappa=1$.

Proof of Theorem 1(iii). By Cauchy–Schwarz, and suppressing n-dependence,

$$\left(\int_A v_\Omega\right)^2\leqslant\left(\int_A d_\Omega^2\right)\left(\int_A\frac{v_\Omega^2}{d_\Omega^2}\right). \tag{31}$$

By (6), the first two equalities in (24), and by (31) we obtain

$$\begin{aligned}\int_\Omega v_\Omega &\geqslant \frac{1}{c_\Omega}\int_A\frac{v_\Omega^2}{d_\Omega^2}\\ &\geqslant \frac{1}{c_\Omega}\frac{\left(\int_A v_\Omega\right)^2}{\int_A d_\Omega^2}.\end{aligned}$$

Hence

$$\int_A v_\Omega\leqslant c_\Omega^{1/2}\left(\int_\Omega v_\Omega\right)^{1/2}\left(\int_A d_\Omega^2\right)^{1/2},$$

and together with (11)

$$\frac{\int_A v_\Omega}{\int_\Omega v_\Omega}\leqslant c_\Omega^{1/2}\frac{\left(\int_A d_\Omega^2\right)^{1/2}}{\left(\int_\Omega v_\Omega\right)^{1/2}}\leqslant(2mc_\Omega)^{1/2}\left(\frac{\int_A d_\Omega^2}{\int_\Omega d_\Omega^2}\right)^{1/2}.$$

This implies the assertion of Theorem 1(iii). □

Proof of Theorem 2(i). We use Theorem 1(i). First note that for a rectangle $R_{a,b}$ with side lengths a and b, where $a\geqslant b$, one has

$$\|d_{R_{a,b}}\|_2^2=\frac{ab^3}{12}-\frac{b^4}{24}\geqslant\frac{ab^3}{24}. \tag{32}$$

To obtain a lower bound for $\|d^2_{\Omega_{\varepsilon_n,n}}\|_1$, we add a closed line segment of length 1 to the boundary of $\Omega_{\varepsilon_n,n}$, and obtain that the resulting set is the union of $n+1$ disjoint rectangles, denoted by $\tilde{\Omega}_{\varepsilon_n,n}$. See Figure 2.

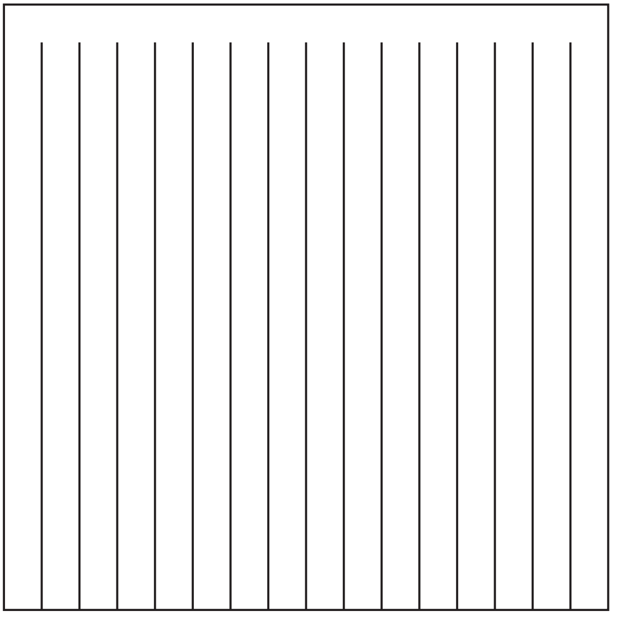

Figure 1: $\Omega_{\frac{1}{16},16}$.

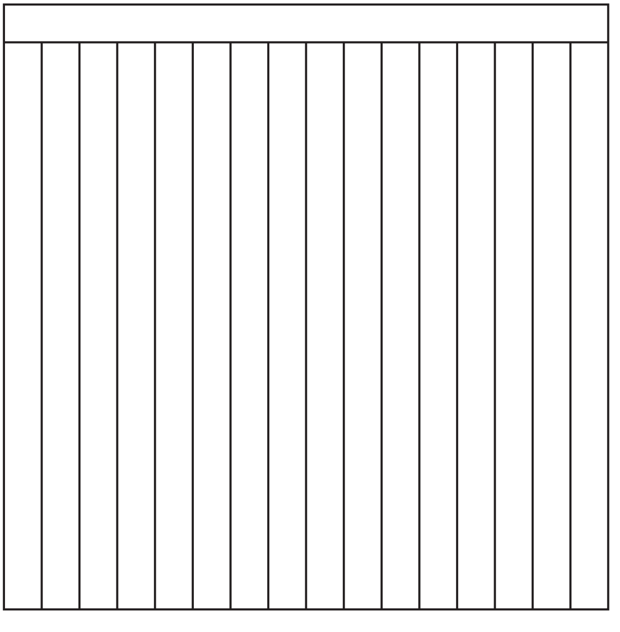

Figure 2: $\tilde{\Omega}_{\frac{1}{16},16}$.

Hence, by (32),

$$\|d^2_{\Omega_{\varepsilon_n,n}}\|_1 \geqslant \|d^2_{\tilde{\Omega}_{\varepsilon_n,n}}\|_1 \geqslant \frac{1-\varepsilon_n}{24n^2} + \frac{\varepsilon_n^3}{24} \geqslant \frac{1}{48}(\varepsilon_n^3 + n^{-2}), \tag{33}$$

where we have used that $\varepsilon_n \leqslant \frac{1}{2}$ for $n \geqslant N_{\alpha,c}$.

To obtain an upper bound for $\|d^2_{\Omega_{\varepsilon_n,n}}\|_1$, we have the following estimate:

$$d_{\Omega_{\varepsilon_n,n}}(x) \leqslant \begin{cases} \frac{1}{2n}, & \forall (x_1,x_2) \in \{\Omega_{\varepsilon_n,n} : x_2 \leqslant 1-\varepsilon_n\}, \\ 1-x_2, & \forall (x_1,x_2) \in \{\Omega_{\varepsilon_n,n} : x_2 \geqslant 1-\varepsilon_n\}. \end{cases} \tag{34}$$

Since

$$|\{\Omega_{\varepsilon_n,n} : x_2 \leqslant 1-\varepsilon_n\}| = 1-\varepsilon_n \leqslant 1, \tag{35}$$

we have by (34) and (35),

$$\int_{\{\Omega_{\varepsilon_n,n}:x_2\leqslant 1-\varepsilon_n\}} d^2_{\Omega_{\varepsilon_n,n}} \leqslant \frac{1}{4n^2} \tag{36}$$

and

$$\int_{\{\Omega_{\varepsilon_n,n}:x_2\geqslant 1-\varepsilon_n\}} d^2_{\Omega_{\varepsilon_n,n}} \leqslant \int_{1-\varepsilon_n}^{1} dx_2 (1-x_2)^2 = \frac{\varepsilon_n^3}{3}. \tag{37}$$

Hence we have

$$\|d^2_{\Omega_{\varepsilon_n,n}}\|_1 \leqslant \frac{1}{3}(\varepsilon_n^3 + n^{-2}). \tag{38}$$

Finally, we observe that

$$\frac{1}{2}\varepsilon_n \leqslant \|d_{\Omega_{\varepsilon_n,n}}\|_\infty \leqslant \varepsilon_n, \qquad n \geqslant N_{\alpha,c}, \tag{39}$$

and (17) follows from (33), (38), (39), Theorem 1(i), $|\Omega_{\varepsilon_n,n}| = 1$, and that for a proper simply connected subset Ω in $\mathbb{R}^2$, $c_\Omega = 16$, [**1**]. This proves the assertion of Theorem 2(i).

Proof of Theorem 2(ii). We use Theorem 1(ii), and choose $\eta_n = \frac{1}{2n}$. Then

$$|\{d_{\Omega_{\varepsilon_n,n}} \geqslant \frac{1}{2n}\}| \leqslant \varepsilon_n,$$

and (15) holds. Since for $n \geqslant N_{\alpha,c}$, $\{d_{\Omega_{\varepsilon_n,n}} \geqslant \frac{1}{2n}\}$ contains a rectangle with side lengths $1-n^{-1}$ and $cn^{-\alpha} - \frac{1}{n}$, respectively, we have by (32)

$$\begin{aligned} \int_{\{d_{\Omega_{\varepsilon_n,n}}\geqslant \frac{1}{2n}\}} d^2_{\Omega_{\varepsilon_n,n}} &\geqslant \frac{1}{24}(1-n^{-1})\big(cn^{-\alpha} - n^{-1}\big)^3 \\ &\geqslant \frac{c^3}{192}(1-n^{-1})n^{-3\alpha} \\ &\geqslant \frac{c^3}{256}n^{-3\alpha}, \end{aligned}$$

where we have used that $cn^{1-\alpha} \geqslant 2$ for $n \geqslant N_{\alpha,c} \geqslant 4$. Hence

$$\frac{\eta_n^2|\Omega_{\varepsilon_n,n}|}{\int_{\{d_{\Omega_{\varepsilon_n,n}}\geqslant\eta_n\}} d^2_{\Omega_{\varepsilon_n,n}}} \leqslant \frac{64n^{3\alpha-2}}{c^3}, \qquad n \geqslant N_{\alpha,c},$$

and (16) holds for $0 < \alpha < \frac{2}{3}$. This proves the assertion of Theorem 2(ii) by Theorem 1(ii).

Proof of Theorem 2(iii). We use Theorem 1(iii), and let for $n \in \mathbb{N}$, A_n be an arbitrary measurable subset of $\Omega_{\varepsilon_n,n}$ such that $\lim_{n\to\infty} |A_n| = 0$. By (33),

$$\|d^2_{\Omega_{\varepsilon_n,n}}\|_1 \geqslant \frac{1}{48n^2}. \tag{40}$$

By (34),

$$\int_{A_n\cap\{x_2\leqslant 1-\varepsilon_n\}} d^2_{\Omega_{\varepsilon_n,n}} \leqslant \frac{|A_n|}{4n^2}. \tag{41}$$

By (37), (40), and (41),

$$\frac{\int_{A_n} d^2_{\Omega_{\varepsilon_n,n}}}{\int_{\Omega_{\varepsilon_n,n}} d^2_{\Omega_{\varepsilon_n,n}}} \leqslant c^3 n^{2-3\alpha} + 12|A_n|.$$

This implies

$$\lim_{n\to\infty} \frac{\int_{A_n} d^2_{\Omega_{\varepsilon_n,n}}}{\int_{\Omega_{\varepsilon_n,n}} d^2_{\Omega_{\varepsilon_n,n}}} = 0,$$

since $\frac{2}{3} < \alpha < 1$, and the hypothesis on A_n. This proves the assertion of Theorem 2(iii) by Theorem 1(iii). □

§3. *Proof of Theorem* 2*(iv)*. Let $(B(s), s \geqslant 0; \mathbb{P}_x, x \in \mathbb{R}^d)$ be Brownian motion associated to the Laplacian in $\mathbb{R}^d$. Here $(B(s), s \geqslant 0)$ takes values in $\mathbb{R}^d$, $\mathbb{P}_x$ is Wiener measure with $\mathbb{P}_x(B(0) = x) = 1$, and for every Borel set $A \in \mathbb{R}^d$,

$$\mathbb{P}_x(B(s) \in A) = (4\pi s)^{-d/2} \int_A dy\, e^{-|x-y|^2/(4s)}, \quad s > 0. \tag{42}$$

If $\Omega \subset \mathbb{R}^d$ is open and $x \in \Omega$, then we denote the first exit time (or life time) of Ω by

$$T_\Omega = \inf\{s \geqslant 0 : B(s) \in \mathbb{R}^2 \setminus \Omega\}.$$

It is convenient to denote the first hitting time of a closed set C by

$$\tau_C = \inf\{s \geqslant 0 : B(s) \in C\}.$$

It is well known that the unique weak solution u_Ω of

$$\Delta u = \frac{\partial u}{\partial t}, \qquad \text{in } \Omega \times (0,\infty),$$

with $u|_{\partial\Omega\times(0,\infty)} = 0$, and with initial datum

$$u(x;0) = 1, \qquad x \in \Omega,$$

is given by

$$u_\Omega(x;t) = \mathbb{P}_x(T_\Omega > t). \tag{43}$$

Since

$$v_\Omega(x) = \int_0^\infty dt\, u_\Omega(x;t), \tag{44}$$

we obtain by (43) and (44) that

$$v_\Omega(x) = \mathbb{E}_x(T_\Omega),$$

and

$$\|v_\Omega\|_1 = \int_\Omega dx \int_0^\infty dt\, u_\Omega(x;t).$$

For the facts above, we refer to [16]. Now let $d = 2$, and let $B = (B_1, B_2)$, where B_1 and B_2 are independent one-dimensional Brownian motions with probability measures $\mathbb{P}^{(1)}_{x_1}$ and $\mathbb{P}^{(1)}_{x_2}$, respectively, and $\mathbb{P}_x = \mathbb{P}^{(1)}_{x_1} \otimes \mathbb{P}^{(2)}_{x_2}$. We have that $\inf\{s \geqslant 0 : B_2(s) = a\} = T_{(-\infty,a)}$. Since

$$\mathbb{P}^{(2)}_{x_2}[T_{(-\infty,a)} > t] = \frac{2}{\pi^{1/2}} \int_0^{(a-x_2)/(4t)^{1/2}} d\theta e^{-\theta^2}, \qquad \forall x_2 < a,$$

we have that the density $\rho(a, \tau)$ of the random variable $\inf\{s \geqslant 0 : B_2(s) = a\}$ with $x_2 = 0$, is given by

$$\rho(a, \tau) = \frac{a}{2\pi^{1/2}\tau^{3/2}} e^{-a^2/(4\tau)} \mathbf{1}_{\mathbb{R}^+}(\tau). \tag{45}$$

The following lemma will be used in the proof of Theorem 2(iv).

LEMMA 3. *Let Ω be an open, bounded, and connected set in $\mathbb{R}^2$ which contains an open rectangle $R_{a,b} \equiv R_{a,b}(p)$, $a \geqslant b$, with sides $K_1 = [(p, 0) - (p + b, 0)]$, $K_2 = [(p, 0) - (p, a)]$, $K_3 = [(p + b, 0) - (p + b, a)]$, $K_4 = [(p + b, a) - (p, a)]$. Let*

$$\Omega_{a,b} \equiv \Omega_{a,b}(p) = \Omega \setminus \left(\cup_{j=1}^{3} K_j\right).$$

If $x \in R_{a,b}$, then

$$v_{\Omega_{a,b}}(x) \leqslant \frac{1}{2}(x_1 - p)(p + b - x_1) + 2^{9/2} e^{-\pi(a-x_2)/(2b)} \lambda_1(\Omega_{a,b})^{-1}. \tag{46}$$

Proof. By (43) and (44), for any $x \in R_{a,b}$,

$$\begin{aligned} v_{\Omega_{a,b}}(x) &= \int_0^\infty dt\, \mathbb{P}_x[T_{\Omega_{a,b}} > t] \\ &= \int_0^\infty dt\, \mathbb{P}_x[T_{R_{a,b}} > t] + \int_0^\infty dt\, \mathbb{P}_x[T_{R_{a,b}} \leqslant t < T_{\Omega_{a,b}}] \\ &\leqslant v_{R_{a,b}}(x_1, x_2) + \int_0^\infty dt\, \mathbb{P}_x[T_{R_{a,b}} \leqslant t < T_{\Omega_{a,b}}] \\ &\leqslant \frac{1}{2}(x_1 - p)(p + b - x_1) + \int_0^\infty dt\, \mathbb{P}_x[T_{R_{a,b}} \leqslant t < T_{\Omega_{a,b}}], \end{aligned} \tag{47}$$

where we have used that $a \mapsto v_{R_{a,b}}(x_1, x_2)$ is monotone increasing and bounded from above by $\frac{1}{2}(x_1 - p)(p + b - x_1)$. See Figure 3. The latter is the torsion function for the interval $(p, p + b)$. To bound the second term in the right-hand side of (47), we use the inclusion

$$\{T_{R_{a,b}} \leqslant t < T_{\Omega_{a,b}}\} \subset \{B(T_{R_{a,b}}) \in K_4\} \cap \{t < T_{\Omega_{a,b}}\}.$$

This inclusion states that the event of a Brownian path starting, for example, at x in $R_{a,b}$ exiting the rectangle $R_{a,b}$ before t but not exiting $\Omega_{a,b}$ before t has to exit the rectangle at K_4, while staying in $\Omega_{a,b}$ until t. By Cauchy–Schwarz,

$$\begin{aligned} \mathbb{P}_x[T_{R_{a,b}} \leqslant t < T_{\Omega_{a,b}}] &\leqslant \left(\mathbb{P}_x[B(T_{R_{a,b}}) \in K_4]\right)^{1/2} (\mathbb{P}_x[t < T_{\Omega_{a,b}}])^{1/2} \\ &\leqslant 2^{3/4} \left(\mathbb{P}_x[B(T_{R_{a,b}}) \in K_4]\right)^{1/2} e^{-t\lambda_1(\Omega_{a,b})/8}, \end{aligned} \tag{48}$$

where we have used that for open sets $\Omega \subset \mathbb{R}^2$, or for open subsets $\Omega \subset \mathbb{R}^1$,

$$\mathbb{P}_x[T_\Omega > t] \leqslant 2^{3/2} e^{-t\lambda_1(\Omega)/4}. \tag{49}$$

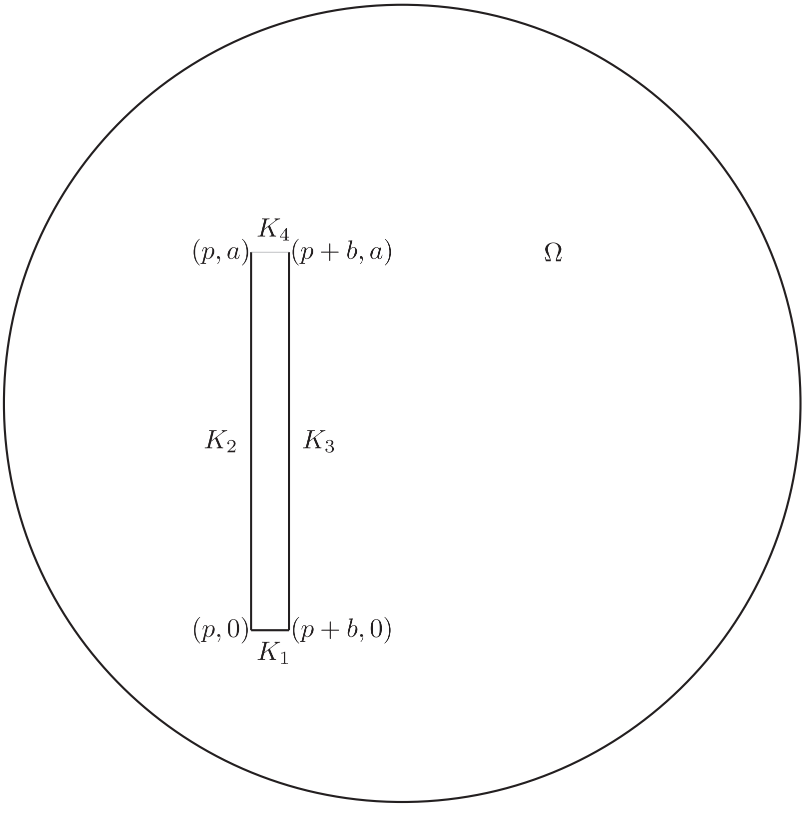

Figure 3: $\Omega_{a,b} = \Omega \setminus (\cup_{j=1}^{3} K_j)$: Rectangle $R_{a,b}$ with sides K_2, K_3 of length a, and K_1, K_4 of length b.

See [**6**, pp. 10, 11]. Let $S_{a,b} = (p, p+b) \times (-\infty, a)$ be the half-strip of width b below the line segment K_4. Note that $R_{a,b} \subset S_{a,b}$ and that $|S_{a,b}| = \infty$. By (48), by applying (49) to the open set $\Omega = (0, b)$, and using (45),

$$\begin{aligned}\mathbb{P}_x[B(T_{R_{a,b}}) \in K_4] &\leqslant \mathbb{P}_x[B(T_{S_{a,b}}) \in K_4] \\ &= \int_0^\infty d\tau\, \rho(a - x_2, \tau)\mathbb{P}_{x_1}^{(1)}[T_{(0,b)} > \tau] \\ &\leqslant 2^{3/2} \int_0^\infty d\tau\, \frac{a - x_2}{2\pi^{1/2}\tau^{3/2}} e^{-(a-x_2)^2/(4\tau) - \pi^2\tau/(4b^2)} \\ &= 2^{3/2} e^{-\pi(a-x_2)/(2b)},\end{aligned} \tag{50}$$

and where we have used [**18**, formula 3.472.3] for the last equality. By (48) and (50)

$$\mathbb{P}_x[T_{R_{a,b}} \leqslant t < T_{\Omega_{a,b}}] \leqslant 2^{3/2} e^{-\pi(a-x_2)/(4b) - t\lambda_1(\Omega_{a,b})/8}. \tag{51}$$

Integrating both sides of (51) with respect to t, and using (47) completes the proof. $\square$

Proof of Theorem 2(iv). To prove (19) we first obtain a lower bound on the right-hand side of (8) by making a particular choice for (A_n):

$$A_n = \{x \in \Omega_{\varepsilon_n, n} : x_2 > 1 - cn^{-2/3}\}. \tag{52}$$

We let

$$B_n = \{x \in \Omega_{\varepsilon_n,n} : 1 - cn^{-2/3} \geqslant x_2 > 1 - cn^{-2/3} - n^{-3/4}\}, \tag{53}$$

and

$$C_n = \{x \in \Omega_{\varepsilon_n,n} : 1 - cn^{-2/3} - n^{-3/4} \geqslant x_2\}, \tag{54}$$

so that $\Omega_{\varepsilon_n,n} = A_n \cup B_n \cup C_n$. By (8), and the monotonicity of the torsion function,

$$\begin{aligned}\kappa &\geqslant \limsup_{n\to\infty} \frac{\int_{A_n} v_{\Omega_{\varepsilon_n,n}}}{\int_{A_n} v_{\Omega_{\varepsilon_n,n}} + \int_{B_n} v_{\Omega_{\varepsilon_n,n}} + \int_{C_n} v_{\Omega_{\varepsilon_n,n}}} \\ &\geqslant \limsup_{n\to\infty} \frac{\int_{A_n} v_{A_n}}{\int_{A_n} v_{A_n} + \int_{B_n} v_{\Omega_{\varepsilon_n,n}} + \int_{C_n} v_{\Omega_{\varepsilon_n,n}}}.\end{aligned} \tag{55}$$

Since A_n is a rectangle with side lengths 1 and $cn^{-2/3}$, respectively, we have by (32)

$$\frac{c^3}{12n^2} \geqslant \int_{A_n} v_{A_n} \geqslant \frac{c^3}{12n^2} - \frac{c^4}{24n^{8/3}}. \tag{56}$$

By (55) and (56),

$$\begin{aligned}\kappa &\geqslant \limsup_{n\to\infty} \frac{c^3 - \frac{1}{2}c^4 n^{-2/3}}{c^3 + 12n^2 \int_{B_n} v_{\Omega_{\varepsilon_n,n}} + 12n^2 \int_{C_n} v_{\Omega_{\varepsilon_n,n}}} \\ &= \limsup_{n\to\infty} \frac{c^3}{c^3 + 12n^2 \int_{B_n} v_{\Omega_{\varepsilon_n,n}} + 12n^2 \int_{C_n} v_{\Omega_{\varepsilon_n,n}}}.\end{aligned} \tag{57}$$

By (21) and (39),

$$\|v_{\Omega_{\varepsilon_n,n}}\|_\infty \leqslant 16\mathfrak{c}_2 \|d_{\Omega_{\varepsilon_n,n}}\|_\infty^2 \leqslant 16\mathfrak{c}_2 c^2 n^{-4/3}, \qquad n \geqslant N_{2/3,c}. \tag{58}$$

Since $|B_n| = n^{-3/4}$, we have by (58),

$$\begin{aligned}12n^2 \int_{B_n} v_{\Omega_{\varepsilon_n,n}} &\leqslant 12n^2 |B_n| \|v_{\Omega_{\varepsilon_n,n}}\|_\infty \\ &\leqslant 192\mathfrak{c}_2 c^2 n^{-1/12}.\end{aligned}$$

This, together with (57), yields

$$\kappa \geqslant \limsup_{n\to\infty} \frac{c^3}{c^3 + 12n^2 \int_{C_n} v_{\Omega_{\varepsilon_n,n}}}. \tag{59}$$

In order to bound the integral in the right-hand side from above, we use Lemma 3 for each of the n rectangles $R_{a,b}(p)$ in $\Omega_{\varepsilon_n,n}$, with $p = \frac{k}{n}$, $0 \leqslant k \leqslant n-1$, with $b = \frac{1}{n}$, and $a = 1 - cn^{-2/3}$. Note that for any $p = \frac{k}{n}$ the set $\Omega_{a,b}(p)$ introduced in Lemma 3 coincides with $\Omega_{\varepsilon_n,n}$, and that each point in C_n satisfies $a - x_2 \geqslant n^{-3/4}$. By (20) and (39),

$$\lambda_1(\Omega_{a,b}) = \lambda_1(\Omega_{\varepsilon_n,n}) \geqslant \frac{1}{16\|d_{\Omega_{\varepsilon_n,n}}\|_\infty^2} \geqslant \frac{1}{16c^2 n^{-4/3}}.$$

Hence the second term in the right-hand side of (46) is bounded from above by $2^{17/2}c^2 n^{-4/3} e^{-\pi n^{1/4}/2}$ uniformly for all points $x \in C_n$. Since $|C_n| \leqslant 1$, we have

$$\int_{C_n} 2^{9/2} e^{-\pi(a-x_2)/(2b)} \lambda_1(\Omega_{a,b})^{-1} \leqslant 2^{17/2} c^2 n^{-4/3} e^{-\pi n^{1/4}/2}. \tag{60}$$

Integrating the first term in the right-hand side of (46) over one rectangle in C_n, and adjusting the coordinate frame appropriately, gives a contribution

$$\int_0^{1-cn^{-2/3}-n^{-3/4}} dx_2 \int_0^{1/n} dx_1 \frac{x_1}{2}(n^{-1}-x_1) = \frac{1}{12n^3}\big(1-cn^{-2/3}-n^{-3/4}\big).$$

Summing over all n rectangles in C_n gives, together with (60) and Lemma 3,

$$12n^2 \int_{C_n} v_{\Omega_{\varepsilon_n,n}} \leqslant 1-cn^{-2/3}-n^{-3/4}+3\cdot 2^{21/2}c^2n^{2/3}e^{-\pi n^{1/4}/2}. \tag{61}$$

By (59) and (61) we conclude

$$\kappa \geqslant \frac{c^3}{1+c^3}.$$

We now prove the converse inequality by obtaining an upper bound for the right-hand side of (8). We first observe, by the monotonicity, that the torsion for $\Omega_{\varepsilon_n,n}$ is bounded from below by the torsion for the set in Figure 2. The latter is the union of n disjoint rectangles $R_{a,b}$ with $b=\frac{1}{n}$ and $a=1-cn^{-2/3}$, together with one disjoint rectangle with $a=1$ and $b=cn^{-2/3}$. By (32) we find

$$\int_{\Omega_{\varepsilon_n,n}} v_{\Omega_{\varepsilon_n,n}} \geqslant \frac{1-cn^{-2/3}}{12n^2}-\frac{1}{24n^3}+\frac{c^3}{12n^2}-\frac{c^4}{24n^{8/3}}. \tag{62}$$

In order to avoid abuse of notation, we keep (52), (53), and (54), and denote by (D_n) an arbitrary sequence of in $\mathfrak{A}((\Omega_{\varepsilon_n,n}))$. That is, $D_n\subset\Omega_{\varepsilon_n,n}$, measurable, $n\in\mathbb{N}$, with $\lim_{n\to\infty}|D_n|=0$. We have

$$\begin{aligned}\int_{D_n} v_{\Omega_{\varepsilon_n,n}} &= \int_{D_n\cap(A_n\cup B_n)} v_{\Omega_{\varepsilon_n,n}} + \int_{D_n\setminus(A_n\cup B_n)} v_{\Omega_{\varepsilon_n,n}}\\ &\leqslant \int_{A_n} v_{\Omega_{\varepsilon_n,n}} + \int_{B_n} v_{\Omega_{\varepsilon_n,n}} + \int_{D_n\cap C_n} v_{\Omega_{\varepsilon_n,n}}.\end{aligned} \tag{63}$$

By (58),

$$\int_{B_n} v_{\Omega_{\varepsilon_n,n}} \leqslant 16\mathfrak{c}_2c^2n^{-25/12}. \tag{64}$$

We use Lemma 3 to bound the third term in the right-hand side of (63) from above. We have, as before, that for each point $x\in C_n$, both $a-x_2\geqslant n^{-3/4}$ and $\frac{x_1}{2}(\frac{1}{n}-x_1)\leqslant\frac{1}{8n^2}$, where the x_1 coordinate is adjusted to the rectangle under consideration. This gives

$$v_{\Omega_{\varepsilon_n,n}}(x) \leqslant \frac{1}{8n^2}+2^{17/2}c^2n^{-4/3}e^{-\pi n^{1/4}/2}, \quad \forall x\in C_n.$$

Hence

$$\begin{aligned}\int_{D_n\cap C_n} v_{\Omega_{\varepsilon_n,n}} &\leqslant |D_n\cap C_n|\left(\frac{1}{8n^2}+2^{17/2}c^2n^{-4/3}e^{-\pi n^{1/4}/2}\right)\\ &\leqslant |D_n|\left(\frac{1}{8n^2}+2^{17/2}c^2n^{-4/3}e^{-\pi n^{1/4}/2}\right).\end{aligned} \tag{65}$$

By (62) and (64),

$$\limsup_{n\to\infty}\frac{\int_{B_n} v_{\Omega_{\varepsilon_n,n}}}{\int_{\Omega_{\varepsilon_n,n}} v_{\Omega_{\varepsilon_n,n}}}=0.$$

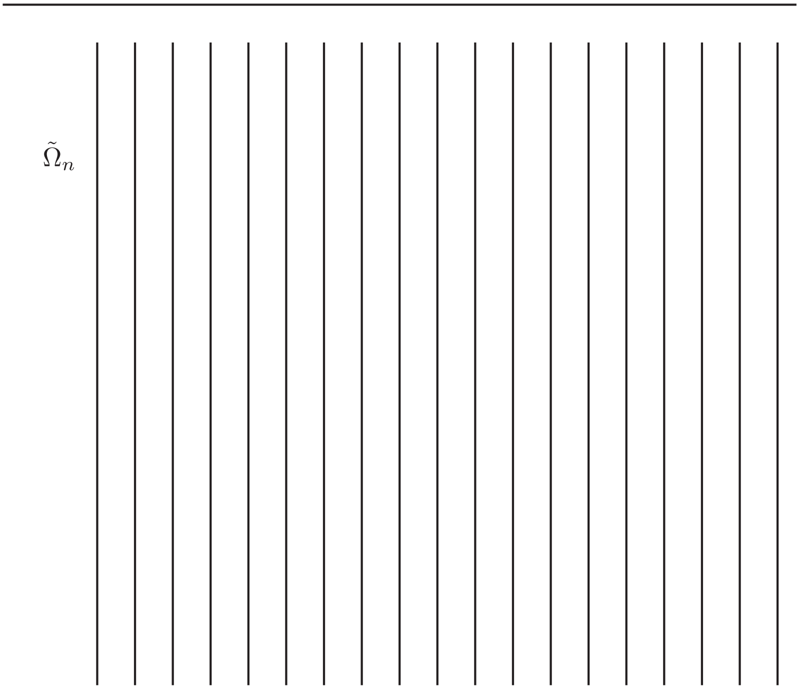

Figure 4: $\tilde{\Omega}_n$: The vertical half lines, periodically extended, are at distance $\frac{1}{n}$, and have distance $cn^{-2/3}$ to the horizontal line.

Moreover, by (62) and (65),

$$\limsup_{n\to\infty} \frac{\int_{D_n\cap C_n} v_{\Omega_{\varepsilon_n,n}}}{\int_{\Omega_{\varepsilon_n,n}} v_{\Omega_{\varepsilon_n,n}}} \leqslant \limsup_{n\to\infty} \frac{4|D_n|}{3(1+c^3)} = 0.$$

It remains to find an upper bound for $\int_{A_n} v_{\Omega_{\varepsilon_n,n}}$. Let $\tilde{\Omega}_n$ be the connected open set in $\mathbb{R}^2$ with boundary consisting of the horizontal line $\{x_2 = 1\}$ and the vertical half-lines $\{x_2 \leqslant 1 - cn^{-2/3}, x_1 = \frac{k}{n},\ k \in \mathbb{Z}\}$. See Figure 4.

We first show that the bottom of the spectrum of the Dirichlet Laplacian on $\tilde{\Omega}_n$ is bounded away from 0. By inserting Neumann boundary conditions on the line $x_2 = 1 - cn^{-2/3}$, the bottom of the spectrum decreases and decouples into the union of the spectra of the horizontal strip at width $cn^{-2/3}$ with Dirichlet boundary conditions on $x_2 = 1$, and Neumann boundary condition on $x_2 = 1 - cn^{-2/3}$, and vertical half-strips of width n^{-1} with Neumann boundary condition at the top end and Dirichlet conditions on the half lines. By taking the double of these half-strips, we obtain that the bottom of the spectrum here is the same as the bottom of the infinite strip of width n^{-1}. Combining these inequalities gives for $n \geqslant N_{\alpha,c}$,

$$\begin{aligned} \lambda_1(\tilde{\Omega}_n) &\geqslant \min\{\frac{\pi^2 n^{4/3}}{4c^2}, \pi^2 n^2\} \\ &= \frac{\pi^2 n^{4/3}}{c^2}. \end{aligned} \tag{66}$$

Hence the torsion function for $\tilde{\Omega}_n$ exists (see (5) and the preceding paragraph), and by the monotonicity property (4)

$$v_{\Omega_{\varepsilon_n,n}}(x) \leqslant v_{\tilde{\Omega}_n}(x), \quad \forall x \in \Omega_{\varepsilon_n,n}.$$

Note that

$$v_{\tilde{\Omega}_n}(x_1, x_2) = v_{\tilde{\Omega}_n}(x_1 + \frac{1}{n}, x_2), \quad \forall x \in \tilde{\Omega}_n.$$

Let

$$E_n = \{x \in \mathbb{R}^2 : 1 > x_2 > 1 - cn^{-2/3} - n^{-3/4}\}.$$

We have

$$\begin{aligned}
\mathbb{P}_x[T_{\Omega_{\varepsilon_n,n}} > t] &\leqslant \mathbb{P}_x[T_{\tilde{\Omega}_n} > t] \\
&= \mathbb{P}_x[T_{\tilde{\Omega}_n} > t,\ T_{E_n} > t] + \mathbb{P}_x[T_{E_n} \leqslant t < T_{\tilde{\Omega}_n}] \\
&\leqslant \mathbb{P}_x[T_{E_n} > t] + \mathbb{P}_x[T_{E_n} \leqslant t < T_{\tilde{\Omega}_n}]. \qquad (67)
\end{aligned}$$

Integrating both sides of (67) with respect to t over $\mathbb{R}^+$ gives

$$\begin{aligned}
v_{\Omega_{\varepsilon_n,n}}(x) &\leqslant v_{E_n}(x) + \int_0^\infty dt\, \mathbb{P}_x[T_{E_n} \leqslant t < T_{\tilde{\Omega}_n}] \\
&= \frac{1}{2}(1 - x_2)(x_2 - 1 + d_n) + \int_0^\infty dt\, \mathbb{P}_x[T_{E_n} \leqslant t < T_{\tilde{\Omega}_n}],
\end{aligned}$$

where

$$d_n = cn^{-2/3} + n^{-3/4}. \qquad (68)$$

Hence

$$\begin{aligned}
\int_{A_n} v_{\Omega_{\varepsilon_n,n}} &\leqslant \int_{A_n} dx\, \frac{1}{2}(1 - x_2)(x_2 - 1 + d_n) + \int_{A_n} dx \int_0^\infty dt\, \mathbb{P}_x[T_{E_n} \leqslant t < T_{\tilde{\Omega}_n}] \\
&\leqslant \frac{1}{12}d_n^3 + \int_{A_n} dx \int_0^\infty dt\, \mathbb{P}_x[T_{E_n} \leqslant t < T_{\tilde{\Omega}_n}].
\end{aligned}$$

By (62) and (68), we have

$$\limsup_{n\to\infty} \frac{\frac{1}{12}d_n^3}{\int_{\Omega_{\varepsilon_n,n}} v_{\Omega_{\varepsilon_n,n}}} \leqslant \frac{c^3}{1 + c^3}.$$

To complete the proof it therefore suffices to show that

$$\limsup_{n\to\infty} n^2 \int_{A_n} dx \int_0^\infty dt\, \mathbb{P}_x[T_{E_n} \leqslant t < T_{\tilde{\Omega}_n}] = 0. \qquad (69)$$

To prove (69) we let $L_n = \{(x_1, 1 - d_n) : x_1 \in \mathbb{R}\}$. We have by the strong Markov property,

$$\begin{aligned}
\mathbb{P}_x[T_{E_n} \leqslant t < T_{\tilde{\Omega}_n}] &\leqslant \mathbb{E}_x\biggl(\int_0^t \mathbf{1}_{\tau_{L_n}\in d\tau} \mathbb{P}_{B(\tau_{L_n})}[T_{\tilde{\Omega}_n} > t - \tau]\biggr) \\
&\leqslant \mathbb{E}_x\biggl(\int_0^t \mathbf{1}_{\tau_{L_n}\in d\tau} \sup_{z\in L_n} \mathbb{P}_z[T_{\tilde{\Omega}_n} > t - \tau]\biggr) \\
&= \int_0^t d\tau \frac{\partial}{\partial\tau}\bigl(\mathbb{P}_x[\tau_{L_n} < \tau]\bigr) \sup_{z\in K_n} \mathbb{P}_z[T_{\tilde{\Omega}_n} > t - \tau]\bigr) \\
&= \int_0^t d\tau \frac{\partial}{\partial\tau}\bigl(\mathbb{P}_x[\tau_{L_n} < \tau]\bigr)\mathbb{P}_{((2n)^{-1}, 1-d_n)}[T_{\tilde{\Omega}_n} > t - \tau], \qquad (70)
\end{aligned}$$

where $\mathbf{1}_{\tau_{L_n}\in d\tau}$ is the indicator function on the set of Brownian paths which hit L_n in the infinitesimal interval $d\tau$. We have used in the final equality in (70) that $x_1 \mapsto \mathbb{P}_{(x_1,1-d_n)}[T_{\tilde{\Omega}_n} > t-\tau]$, $x_1 \in \mathbb{R}$ is periodic in x_1 with period n^{-1}, and which has equal maxima at $\{((2n)^{-1} + kn^{-1}, 1-d_n) : k \in \mathbb{Z}\}$. Hence the supremum in the third line in (70) is a maximum for $z = ((2n)^{-1}, 1-d_n)$. Integrating the convolution in the right-hand side of (70) with respect to t over $\mathbb{R}^+$ gives

$$\begin{aligned}\int_0^\infty dt\, \mathbb{P}_x[T_{E_n} \leqslant t < T_{\tilde{\Omega}_n}] \\ &\leqslant \left(\int_0^\infty d\tau \frac{\partial}{\partial\tau}\big(\mathbb{P}_x[\tau_{L_n} < \tau]\big)\right)\int_0^\infty d\tau\, \mathbb{P}_{((2n)^{-1},1-d_n)}[T_{\tilde{\Omega}_n} > \tau] \\ &= v_{\tilde{\Omega}_n}((2n)^{-1}, 1-d_n).\end{aligned}$$

One verifies that Lemma 3 also holds if Ω contains the open rectangle $(p, p+b) \times (-\infty, a)$. Indeed, the first term in the right-hand side of (46) is the torsion function for the infinite strip of width b. The second term takes into account that the strip is cutoff at a. With this we arrive at

$$\int_0^t dt\, \mathbb{P}_x[T_{E_n} \leqslant t < T_{\tilde{\Omega}_n}] \leqslant \frac{1}{8n^2} + 2^{9/2} e^{-\pi n^{1/4}/2} \lambda_1(\tilde{\Omega}_n)^{-1}. \tag{71}$$

Since $|A_n| = cn^{-2/3}$, we find by (71) and (66),

$$n^2 \int_{A_n} dx \int_0^t dt\, \mathbb{P}_x[T_{E_n} \leqslant t < T_{\tilde{\Omega}_n}] \leqslant \frac{c}{8n^{2/3}} + 2^{9/2}\pi^{-2}c^3 e^{-\pi n^{1/4}/2}.$$

This implies (69). □

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